Euler-Bernoulli and Timoshenko Beam Theories Analytical and Numerical Comprehensive Revision

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Abstract — Obtaining reliable and efficient results of a specified problem solution depends upon understanding the strategy of the method of analysis, which is emanated from all related *physical basics* of the problem, formulated with master mathematical tools to give its governing mathematical model. These two categories require deep study in a wide range of references and literature in order not only to apply the method professionally, but also to look for improvements, developments, and contributions in the field of the method. Consequently, although Euler-Bernoulli and Timoshenko beam theories are the oldest ones, but surely, they represent a cornerstone for most modern methods in structural analysis; In what follows, a detailed revision of these theories and their applications analytically and in numerical style is presented in a proper and simplified entrance to be able to understand more advanced topics such as thin and thick plate theories. Illustrative examples will be used to show and discuss the methods.

Key words — Euler-Bernoulli Beam Theory, First order Shear Deformation; Timoshenko, Beam Theory.

I. INTRODUCTION

Most practical engineering structures consist of structural members such as beams, plates, and shells in which a beam member have a large ratio of length to lateral dimensions and is subjected to forces both along and transverse to its length as well as moments that tend to rotate the member about an axis perpendicular to its length. The corresponding unknowns depend only on the beam-axis position and its structural behavior governed by the well-known beam theories, which represent the cornerstone of more modern analysis theories for many other structural members. The beam theories, in case of small deflections, are classified into two main groups. The first ignores the shear deformations whereas the other takes this effect into account.

In this article a grasp simplified revision of the fundamentals for such important foundational beam theories is presented. These beam theories have great importance in plate and shell studies since beams are counterpart of plates. Furthermore, beam theories enhance great value for understanding the formulating process of the finite elements of thin and thick structural members.

II. EULER-BEMOULLI BEAM THEORY

Due to the nature of beams in structural engineering, it is more convenient to investigate a single spanned, prismatic

constant cross sectional homogenous beam in Cartesian right-handed 3D-coordinate axes system, such that, the positive x is left to right direction and positive y is directed toward the viewer and positive z is in a downward direction. All terminologies and signed quantities in whole of this article will be referenced to this coordinate system which is also the most used one in analytical and numerical plate theories and in finite element methods. Moreover, in order to clarify the basic fundamentals of such beam analysis, it is better to consider as an illustrative entrance, the real deflection curve for a built-in beam subjected to uniform load as shown in Fig.1(a), in which two inflection points divided the beam into three parts. The outer parts concaved- down while the middle part concaved- up. Points 1,2,3 and 4 are general points at which the perpendiculars to the tangents (i.e., *normals*) are shown. These portions are rearranged as shown in Fig.1(b) for mathematical purposes. This deflection curve was obtained based on kinematical assumptions due to Euler/Bernoulli-Navier hypothesis, which in turn leads to the well-known Euler-Bemoulli Beam Theory EBBT. The relevant basic data and relations concerned with Euler-Bemoulli Beam **EBB** which are followed in many important references such as [1]-[5] are summarized in the Table I. The most often used sign convention in main textbooks for analysis quantities such as shown in Fig. 1(a), as well as the mathematical definition of positive orientation of surfaces were followed in this article.

Fig. 1(a). Exaggerated deflection curve for a built-in beam.

Fig. 1(b). Concaved-up and concaved-down portions.

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TABLE I: EULER-BEMOULLI BEAM THEORY

Outer parts of deflection curve
Sign Convention
Concave	down
Variation of θ	from -ve to +ve quantity $\Rightarrow \theta$; increasing $\Rightarrow$ positive
Curvature $\kappa = \left \frac{d\theta}{ds} \right $	$\kappa \stackrel{\text{def}}{=} \frac{1}{\rho^+} = \frac{d\theta^+}{dx} = \frac{d^2w^+}{dx^2} \rightarrow \frac{1}{\rho} = \frac{d\theta}{dx} = \frac{d^2w}{dx^2}$
Local $u_{xx}(x)$	left: $-u_x^u(x, z) = (-z)(-\theta) \rightarrow u_x = -z\theta$: shortening right: $+u_x^u(x, z) = (-z)(+\theta) \rightarrow u_x = -z\theta$: extension left: $+u_x^l(x, z) = (+z)(-\theta) \rightarrow u_x = -z\theta$: extension right: $-u_x^l(x, z) = (+z)(+\theta) \rightarrow u_x = -z\theta$: shortening
Middle part of deflection curve
Sign Convention
Concave	up
Variation of θ	from +ve to -ve quantity $\Rightarrow \theta$; decreasing $\Rightarrow$ negative
Curvature $\kappa = \left \frac{d\theta}{ds} \right $	$\kappa \stackrel{\text{def}}{=} \frac{1}{\rho^-} = \frac{d\theta^-}{dx} = \frac{d^2w^-}{dx^2} \rightarrow \frac{1}{\rho} = -\frac{d\theta}{dx} = -\frac{d^2w}{dx^2}$
Local $u_{xx}(x)$	left: $+u_x^u(x, z) = (-z)(+\theta) \rightarrow u_x = -z\theta$: extension right: $-u_x^u(x, z) = (-z)(-\theta) \rightarrow u_x = -z\theta$: shortening left: $-u_x^l(x, z) = (+z)(+\theta) \rightarrow u_x = -z\theta$: shortening right: $+u_x^l(x, z) = (+z)(-\theta) \rightarrow u_x = -z\theta$: extension
SUMMARY OF RELATIONS	
Generalized u_{xx}	$u_{xx} = -z\theta \equiv -z \frac{dw}{dx}$
$\epsilon_{xx}(x, z)$	$\epsilon_{xx} = \frac{du_x}{dx} = -z \frac{d\theta}{dx} = -z \frac{d^2w}{dx^2}$
$\sigma_{xx}(x, z)$	$\sigma_{xx} = E\epsilon_{xx} = -zE \frac{d\theta}{dx} = -zE \frac{d^2w}{dx^2}$
$M_{xx}(x)$	$M_{xx} = \iint_{A_{yz}} \sigma_{xx}(x, z) z dA_{yz} = -EI \frac{d\theta}{dx} = -EI \frac{d^2w}{dx^2}$
Equilibrium Equations	$\frac{dV}{dx} = -q, \frac{dM_x}{dx} = V$
Derived Relations	$M_{xx} = -EI \frac{d\theta}{dx} = -EI \frac{d^2w}{dx^2}$ $\frac{dM_{xx}}{dx} = V_{xx} = -EI \frac{d^3w}{dx^3}$ $\frac{d^2M_{xx}}{dx^2} \equiv \frac{dV_{xx}}{dx} \equiv -EI \frac{d^4w}{dx^4} = -q(x)$
Governing Differential Equation	$EI \frac{d^4w}{dx^4} - q(x) = 0, 0 \leq x \leq L$

III. SHEARING STRESSES & STRAINS IN BEAMS

Practically, in a non-uniform bending moment case, both shear and bending coexist at the beam section with $dM/dx = V \neq 0$ which is different from the case of pure bending. The

vertical shear force V acting tangentially upon the plane of a rectangular cross section having width b , thickness t , area $A = bt$ and moment of inertia $I = bt^3/12$, produces an average shear stress $\tau_{xz}^{av} = V/A$, which is constant throughout the entire section. However, the actual shear stress τ_{xz}^{ac} as introduced by Timoshenko [1] is given by:

$$\tau_{xz}^{ac} \equiv \tau_{xz} = \frac{V}{2I} \left(\frac{t^2}{4} - z^2 \right) \equiv \frac{6V}{At^2} \left(\frac{t^2}{4} - z^2 \right)$$

which follows a parabolic law and vanishes at free surfaces $z = \pm t/2$ as it must be since the complementary shear stresses are zero. This stress has a maximum value at points that lies on the neutral axis (i.e., $z = 0$) of the cross section; that is: $\max. \tau_{xz}^{ac} = \frac{3}{2} \frac{V}{bt} \equiv \frac{3}{2} \tau_{xz}^{av}$. Such structural beam members, which account for the shearing effect are known as **Timoshenko Beam** and are governed by **Timoshenko Beam theory (TBT)**. Such shear stress distribution causes the initially plane cross section to become **warped** (i.e., curved), but it remains orthogonal to the upper and lower surfaces, furthermore Timoshenko [1] showed that this **warp** is the same as long as the shearing force remains constant or continuously varying along the beam, in which the adjacent sides of infinitesimal sectional element slides to each other [1], by same amount of displacement in either sides of bent portion, provided no elongation or shortening of the longitudinal beam fibers occur.

In order to show the real distortion of such warped sections diagrammatically, it is more helpful to use the mathematical solutions based on theory of elasticity as given by Popov [4], Timoshenko [6] and later on by Reddy [7]. The 2D-Elasticity solution introduced by Timoshenko for a rectangular cross sectional cantilever beam of length L subjected to point load P applied at its free end leads to two expressions for longitudinal and transversal (i.e., deflection) displacements; $u(x, z)$ & $w(x, z)$ respectively. The first one leads to a deflection curve that exactly coincides with the one obtained by **EBBT** due to constraint condition $\partial w / \partial x = 0$ at the left fixed end, whereas the second expression is based on the constraint condition $\partial u / \partial y = 0$ at fixed end of the beam. These two expressions are:

$$u(x, z) = -\frac{Px^2z}{2EI} - \frac{vPz^3}{6EI} + \frac{PL^2z}{2EI} - \frac{Pz^3}{6IG}$$

$$w(x, z) = \frac{vPxz^2}{2EI} + \frac{Px^3}{6EI} - \frac{PL^2x}{2EI} + \frac{PL^3}{3EI} + \frac{PLc^2}{2IG} - \frac{Pxc^2}{2IG}$$

where $c = t/2$, E modulus of elasticity and G shear modulus, which show the contribution of shear effect to the displacements. Substitution of $z = \pm t/2$ in these 2D-equations, instead of giving the deflected longitudinal fibers within the beam, it will draw only the deflection curve of the longitudinal beam axis (i.e., at $z = 0$), which is:

$$w(x) = \frac{Px^3}{6EI} - \frac{PL^2x}{2EI} + \frac{PL^3}{3EI} + \frac{PLc^2}{2IG} - \frac{Pxc^2}{2IG}$$

equivalently, with a specified constant $K_s = 2/3$ as:

$$w(x) = \frac{Px^3}{6EI} - \frac{PL^2x}{2EI} + \frac{PL^3}{3EI} + \frac{P}{K_s GA} (L - x)$$

The 3D-Elasticity solution introduced by Reddy for the same cantilever beam leads to the same last deflection equation $w(x)$ for small t/L ratio but with $K_s = 4/3$.

Along thickness-wise (i.e., at any z), the longitudinal displacement function $u(x, z)$ at any x along span length will draw the warped sections, these distorted sections are determined for $x = L$ and plotted exactly using Maple package, in Fig. 2(a) & Fig. 2(b) for both 2D- & 3D-solutions which in turn are shows a non-usual condition at built-in end since practically, the fixed support is not free to distort [6].

Fig. 2. Exaggerated warpage shapes.

The deformed beam is also shown in Fig. 3 based on the last $w(x)$ and different sections deduced by Timoshenko $u(x)$ -function.

Fig. 3. Deformed cantilever beam (warped cross sections).

In case of *pure shear*, the angle between each of tangents to each of these warped curves and their corresponding initial normal Fig. 2(c) is equal to the shearing strain γ provided that its maximum value occurs at neutral axis as natural consequence of shearing stress distribution effect, which in turn was assigned by Timoshenko for the whole of each section although it must vary in the same manner as shearing stress. Consequently, this theory relaxes the *Bernoulli-Navier normality hypothesis*, but at the same time neglects the warpage shape of originally plane cross section, therefore this assumed *constant shearing strain* through the beam thickness should correspond to a constant shearing stress $\max. \tau_{xz}^{ac} = (3/2)(V/bh) \equiv 3/2 \tau_{xz}^{av}$. Therefore, the pertinent constitutive Hookean relationship gives:

$$(\max. \gamma_{xz}^{ac}) = \frac{(\max. \tau_{xz}^{ac})}{G} = \frac{\tau_{xz}^{av}}{(2/3) G}$$

In general, $\gamma_{xz} = V/K_s GA$, where the so-called shear correction factor or shear stiffness K_s is introduced, to compensate for or *correct* the error due to the foregoing assumption. This factor as introduced in first time by Timoshenko in 1920/1921; was 2/3 for static analysis [8] and 8/9 for dynamic analysis [9] has been the subject of many investigations and many other approximations now exist, such as Roark's values [10] which are based on the work of

Newlin and Trayer [11] for rectangular cross-sectional beam as $K_s = 5/6$. A more accurate value was given by Cowper [12] as: $K_s = 10(1 + \nu)/(12 + 11\nu)$, which is independent of the aspect ratio of the rectangular cross section, and also agrees with Timoshenko's and Roark's values for $\nu = 0$ [13].

As a closure, since (TBT) assumes a deformed **straight** section and neglecting the **warpage** which may be described by a higher degree (order) polynomial of z , then recently this theory is also, well and wide known as **First order Shear Deformation Theory (FSDT)**

IV. TIMOSHENKO BEAM THEORY

The deformation of elements of Timoshenko beam undergoes a rotation ϕ of the beam cross-section due to bending, in which as will be seen later in illustrative examples this rotation is independent of shear effect for all statically determinate beams and also all indeterminate beams having full symmetric conditions whereas except these beam-types, this rotation will depend on shear stiffness $K_s GA$. the bending moment M_x will behave in the same way. Therefore, the rotation ϕ in general will not be the same as in the Euler-Bernoulli beam. That is; $\phi \neq \theta \equiv dw^E(x)/dx$. Moreover, the deformation of such elements also undergoes an additional *angular rotation* φ caused by shear. Thereby the **total rotation** of the Normal is the slope of the deflection curve w^T that is; $dw^T(x)/dx$ and numerically is equal the *sum of these two rotations* as $dw^T(x)/dx = (\phi + \varphi)$, where the superscripts E & T refer to Euler and Timoshenko theories respectively.

The previous mathematical elasticity solutions can also be utilized to emphasize such sum, which follows what was visualized by Timoshenko [8] and others [14]-[17], where the visualization appeared by the almost similar figures; 12.1, 3.4.1, 10.1.1 and 8.27 given in Reddy *et al.* [18]-[21] respectively, is redrawn as shown in Fig. 4:

Fig. 4. Deformation of cross section described by Reddy.

This figure shows that the *total rotation* of the normal denoted ϕ_x was utilized instead of $dw^T(x)/dx$, as a sum of *angular rotation* $\gamma_{xz} \equiv \varphi$ and the rotation $-\partial w_0(x)/\partial x$ in which the later one maintains *Bernoulli-Navier normality hypothesis* (i.e., *section remains normal*) as seen in Reddy-figures (i.e., Fig. 4). Thus, one may understand that $w_0(x) \equiv w^E$, in which this $w_0(x)$ becomes an essential part in describing the shearing angular strain γ_{xz} as:

$$\gamma_{xz} = \phi_x - \left(-\frac{\partial w_0}{\partial x}\right) = \frac{\partial w_0}{\partial x} + \phi_x$$

Consequently, Reddy included shear force in the governing equilibrium equations (2.5.33, [7]) & (2.5.34, [7]) and also equations (4.3.10 a & b, [19]) which are:

$$-\frac{d}{dx} \left[K_s GA \left(\frac{dw_0}{dx} + \phi(x) \right) \right] = q$$

$$K_s GA \left(\frac{dw_0}{dx} + \phi(x) \right) - \frac{d}{dx} \left(EI \frac{d\phi}{dx} \right) = 0$$

to determine $w_0(x)$ & $\phi(x)$ and the final solutions of Reddy-examples 4.3.1 and 4.3.2 in Ref. [19] showed that the deflection w^T is the one which was obtained.

These last two equations were also given in Wang [22], by his Eqs. (2.1.31) & (2.1.32) but at this time the superscript T denoting to Timoshenko is added. These two γ_{xz} – expressions will be discussed later in the light of the obtained analytical solutions of incoming illustrative examples. Therewith the *total rotation* described by Timoshenko [8] as discussed earlier will be followed hereinafter. Consequently, in similar manner to elementary **EBBT**, the bending longitudinal displacement $u(x, z) = -z \phi(x)$, i.e., *not affected by shears*, and then bending curvature is $\kappa = -d\phi/dx$, because of $\phi(x)$ decreases vs. increases in arc length ds of deflected curve then the previously moment-curvature equation becomes as given by Timoshenko [8]:

$$M(x) = -EI \frac{d\phi(x)}{dx} \quad (1)$$

The equilibrium equation $V(x) = dM(x)/dx$ yields:

$$V(x) = -EI \frac{d^2\phi(x)}{dx^2} \quad (2)$$

Therefore, from the well-known definition of two-dimensional strain field in terms of displacements, the *engineering shearing strain* is given by:

$$\gamma_{xy} = \frac{dw^T}{dx} + \frac{du}{dz} = \frac{dw^T}{dx} - \phi,$$

where it is evident that a direct substitution for earlier dw^T/dx satisfies the aforementioned statement that; the *angular rotation* ϕ represents the shearing strain, thereby alternatively, the shear force (2) can be expressed as:

$$V(x) = K_s GA \left(\frac{dw^T(x)}{dx} - \phi(x) \right) \quad (3)$$

Definitely in cases of statically determinate beams in which $M(x)$ & $V(x)$ can be expressed in terms of known applied loads, then $\phi(x)$ and $w(x)$ can be easily obtained by means of (1) & (2) respectively, while in other beam problems, the following derived relationships provide the way for solution:

- Equality of (2) and (3) yields:

$$EI \frac{d^2\phi}{dx^2} + K_s GA \left(\frac{dw^T(x)}{dx} - \phi(x) \right) = 0 \quad (4)$$

- From (2) and (3), the equilibrium equation $\frac{dV(x)}{dx} = -q(x)$ can be re-expressed as:

$$\frac{dV(x)}{dx} = \left\{ \begin{array}{l} -EI \frac{d^3\phi(x)}{dx^3} \\ K_s GA \left(\frac{d^2w^T(x)}{dx^2} - \frac{d\phi(x)}{dx} \right) \end{array} \right\} = -q(x) \quad (5)$$

- From which the following two equations can easily be concluded:

$$EI \frac{d^3\phi(x)}{dx^3} = q(x) \quad (6)$$

$$\frac{d^2w^T(x)}{dx^2} = \frac{d\phi(x)}{dx} - \frac{q(x)}{K_s GA} \quad (7)$$

- Equality of middle parts in (5) gives:

$$EI \frac{d^3\phi(x)}{dx^3} + K_s GA \left(\frac{d^2w^T(x)}{dx^2} - \frac{d\phi(x)}{dx} \right) = 0 \quad (8)$$

- Furthermore, twice differentiation of (7) is:

$$\frac{d^4w^T(x)}{dx^4} = \frac{d^3\phi(x)}{dx^3} - \frac{1}{K_s GA} \frac{d^2}{dx^2} q(x) \quad (9)$$

- Then from (6) into (9) yields:

$$\frac{d^4w^T(x)}{dx^4} = \frac{q(x)}{EI} - \frac{1}{K_s GA} \frac{d^2}{dx^2} q(x) \quad (10)$$

which is the fourth order linear differential equation governing the deflection of Timoshenko beam, It is evident that the analytical solution of (10) requires the application of four boundary conditions (BC's) which may give rise to some complexity, Therefore our proposed systemized routine is that, the solution of such problem can be simplified by solving (6) with its BC's for ϕ , then a direct evaluation of (7) with its BC's for w , and the complete solution for ϕ & w , can be obtained through determination of remaining arbitrary constants by using of *automatic relation* (4), as will be illustrated in several examples later. But in the following section a numerical solution for both Euler-Bernoulli as well as Timoshenko beams using finite element method with mixed formulation is also investigated for completion of beam analysis alternatives as compared with analytical solutions. Moreover, such investigation provides a simple foundational entrance for interested beginners in more advanced topics.

V. MIXED FINITE BEAM ELEMENT MODELS

Finite element formulations in elasticity provides wide strategies to tackle many kinds of problems, usually based on multi-field variational principles in the form of weighted-residual integrals as well as their weaken forms in which the

mixing between these two strong and weak forms nowadays is often known as primal-mixed and Dual-mixed formulation. A valuable study of such topic is given in [23], in what follows the weak formulation will be followed.

A. Euler-Bernoulli beam Mixed Finite Model

In order to maintain the symmetry as it will be seen later, the equilibrium governing equations can be rewritten as:

$$M_x = -EI \frac{d^2 w}{dx^2} \rightarrow -\frac{d^2 w}{dx^2} - \frac{M_x}{EI} = 0$$

$$\frac{dM_x}{dx} = V \text{ \& } \frac{dV}{dx} = -q(x) \rightarrow -\frac{d^2 M_x}{dx^2} - q(x) = 0$$

The mixed finite element model can be constructed by applying either of both equivalently: three steps procedure due to Reddy [21], or weak form approach described by Zienkiewicz in Ref. [15]. Over an element $\Omega^e = (x_a, x_b) \equiv (0, h)$ whose fixed length is $h = x_b - x_a$, the following weighted-residual integrals:

$$\int_0^h \delta M_x \left(-\frac{M_x}{EI} - \frac{d^2 w}{dx^2} \right) dx = 0$$

$$\int_0^h \delta w \left(-\frac{d^2 M_x}{dx^2} - q(x) \right) dx = 0$$

are used. A simplified detailed discussion of weakening such integrals can be found in first-author *et-al.* previous works [24]-[27], then the weak forms of these governing equations are:

$$-\frac{1}{EI} \int_0^h \delta M_x M_x dx + \int_0^h \frac{d\delta M_x}{dx} \frac{dw}{dx} dx - \left[\delta M_x \frac{dw}{dx} \right]_0^h = 0$$

$$\int_0^h \frac{d\delta w}{dx} \frac{dM_x}{dx} dx - \int_0^h \delta w q(x) dx - \left[\delta w \frac{dM_x}{dx} \right]_0^h = 0$$

which suggests linear interpolation functions for each of the field variables M_x , and w , in terms of deflection and moment at nodal values w_a, w_b & m_{xa}, m_{xb} as:

$$w(x) = \left(1 - \frac{x}{h}\right) w_a + \left(\frac{x}{h}\right) w_b$$

$$M(x) = \left(1 - \frac{x}{h}\right) m_{xa} + \left(\frac{x}{h}\right) m_{xb}$$

or in augmented matrix form, (where the superscript T over matrix reads transpose), as:

$$w(x) = \begin{bmatrix} 0 & 1 - \frac{x}{h} & 0 & \frac{x}{h} \end{bmatrix} \begin{bmatrix} 0 & w_a & 0 & w_b \end{bmatrix}^T$$

$$M_x(x) = \begin{bmatrix} 1 - \frac{x}{h} & 0 & \frac{x}{h} & 0 \end{bmatrix} \begin{bmatrix} m_{xa} & 0 & m_{xb} & 0 \end{bmatrix}^T$$

Substitution of such interpolations into weak form yields the following finite element model:

$$\begin{bmatrix} [K_{11}] & [K_{12}] \\ [K_{21}] & [K_{22}] \end{bmatrix} \begin{bmatrix} [M_x^e] \\ [w^e] \end{bmatrix} = \begin{bmatrix} [F_1^e] \\ [F_2^e] \end{bmatrix} + \begin{bmatrix} [F_3^e] \\ [F_4^e] \end{bmatrix}$$

$$[K_{11}] = -\frac{1}{EI} \int_0^h [N_{Mx}]^T [N_{Mx}] dx$$

$$[K_{12}] = \int_0^h \frac{d[N_{Mx}]^T}{dx} \frac{d[N_w]}{dx} dx$$

$$[K_{12}] = \int_0^h \frac{d[N_w]^T}{dx} \frac{d[N_{Mx}]}{dx} dx, [K_{22}] = [0]$$

$$[F_2^e] = \int_0^h [N_w]^T q_0 dx = \begin{bmatrix} 0 & \frac{q_0 h}{2} & 0 & \frac{q_0 h}{2} \end{bmatrix}$$

q_0 is the intensity of uniformly distributed load.

$$\theta_a = (dw/dx)_a, \theta_b = (dw/dx)_b$$

$$V_a = (dM_x/dx)_a, V_b = (dM_x/dx)_b$$

$$[F_4^e] = [V] = \begin{bmatrix} V_a & 0 & V_b & 0 \end{bmatrix}$$

$$[F_3^e] = [\Theta] = \begin{bmatrix} 0 & \theta_a & 0 & \theta_b \end{bmatrix}$$

Evaluations of internal matrix elements $[K_{ij}]$ yields the following augmented finite element matrix:

$$\begin{bmatrix} -\frac{h}{3EI} & \frac{1}{h} & -\frac{h}{6EI} & -\frac{1}{h} \\ \frac{1}{h} & 0 & -\frac{1}{h} & 0 \\ -\frac{h}{6EI} & -\frac{1}{h} & \frac{h}{3EI} & \frac{1}{h} \\ -\frac{1}{h} & 0 & \frac{1}{h} & 0 \end{bmatrix} \begin{Bmatrix} mx_a \\ w_a \\ mx_b \\ w_b \end{Bmatrix} = \begin{Bmatrix} 0 \\ \frac{qh}{2} \\ 0 \\ \frac{qh}{2} \end{Bmatrix} + \begin{Bmatrix} V_a \\ \theta_a \\ V_b \\ \theta_b \end{Bmatrix}$$

or in short form: $[K^e][U^e] = [F^e] + [\bar{F}^e]$

Due to inter-element continuity and also when w or M_x are prescribed at beam ends we have $\delta M_x = 0$ and $\delta w = 0$. Therefore, $[F_4^e]$ & $[F_3^e]$ vanish in overall augmented matrix i.e., $[\bar{F}^e] = 0$. Moreover, for uniformly distributed loaded beam with homogeneous boundary condition, $[F_1^e] = [0]$, hence the last equation becomes as: $[K^e][U^e] = [F^e]$, where $[U^e] = [mx_a \ w_a \ mx_b \ w_b]^T$.

B. Timoshenko beam Mixed Finite Model

Similar to Euler-Bernoulli beam; utilizing equilibrium equations and their relevant basic relations (1) and (3), in which the superscript T that is referred to Timoshenko as well as the functionality (x) except for $q(x)$ will be omitted here for sake of space. Then over a typical element Ω^e between nodes 1 and 2, the first equilibrium equation:

$$\frac{dV}{dx} = -q(x) \rightarrow -\frac{dV}{dx} - q(x) = 0$$

its weighted-residual integral is:

$$-\int_0^h \delta w \frac{dV}{dx} dx - \int_0^h \delta w q(x) dx = 0$$

Weakening of the first integral yields:

$$\int_0^h \frac{d\delta w}{dx} V dx - \int_0^h \delta w q(x) dx - \delta w \bar{V}|_1^2 = 0 \quad (11)$$

The second form of the shear force basic relation i.e., (3):

$$V = K_s GA \left(\frac{dw}{dx} - \phi \right),$$

giving

$$K_s GA \left(\frac{dw}{dx} - \phi \right) - V = 0$$

with weighted residual-integral

$$\int_0^h \delta V \frac{dw}{dx} dx - \int_0^h \delta V \phi dx - \int_0^h \delta V \frac{V}{K_s GA} dx = 0$$

and the weaken process yields

$$-\int_0^h \frac{d\delta V}{dx} w dx - \int_0^h \delta V \phi dx - \int_0^h \delta V \frac{V}{K_s GA} dx + \delta V \bar{w}|_1^2 = 0 \quad (12)$$

Observing that the first integral in (12) does not conform in symmetric condition with its corresponding one in (11), and for the purpose of retaining the symmetry, it is required to repeat the integration in the form:

$$\int_0^h \delta V \frac{dw}{dx} dx - \int_0^h \delta V \phi dx - \int_0^h \delta V \frac{V}{K_s GA} dx + \delta V (\bar{w} - w)|_1^2 = 0 \quad (12)^*$$

Similarly, for second equilibrium equation:

$$\frac{dM}{dx} = V \rightarrow -\frac{dM}{dx} + V = 0$$

Its weighted residual integral is:

$$-\int_0^h \delta \phi \frac{dM}{dx} dx + \int_0^h \delta \phi V dx = 0$$

and after weakening of its first part, becomes:

$$\int_0^h \frac{d\delta \phi}{dx} M dx + \int_0^h \delta \phi V dx - \delta \phi \bar{M}|_1^2 = 0 \quad (13)$$

The corresponding basic relation:

$$M = -EI \frac{d\phi(x)}{dx} \rightarrow -\frac{M}{EI} - \frac{d\phi}{dx} = 0$$

and its weighted-residual integral is:

$$-\int_0^h \delta M \frac{d\phi}{dx} dx - \int_0^h \delta M \frac{M}{EI} dx = 0$$

Weakening this first integral yields:

$$\int_0^h \frac{d\delta M}{dx} \phi dx - \int_0^h \delta M \frac{M}{EI} dx - \delta M \bar{\phi}|_1^2 = 0$$

Retaining the conformity with (13) in conditions of symmetry for the first integral as:

$$-\int_0^h \delta M \frac{d\phi}{dx} dx - \int_0^h \delta M \frac{M}{EI} dx + \delta M (\phi - \bar{\phi})|_1^2 = 0 \quad (14)$$

Finally multiplying both sides of (13) by (-1) yielding:

$$-\int_0^h \frac{d\delta \phi}{dx} M dx - \int_0^h \delta \phi V dx + \delta \phi \bar{M}|_1^2 = 0 \quad (13)^*$$

The weak forms (11), (12)*, (13)* and (14) provide the following mixed finite element augmented *symmetrical* matrix and loading vector for Timoshenko beam when all boundary terms are omitted:

$$[K^e] = \int_0^h \begin{bmatrix} \left[-M \frac{M}{EI} \right] & [0] & \left[-M \frac{d\phi}{dx} \right] & [0] \\ [0] & \left[-V \frac{V}{K_s GA} \right] & [-V \phi] & \left[V \frac{dw}{dx} \right] \\ \left[-\frac{d\phi}{dx} M \right] & [-\phi V] & [0] & [0] \\ [0] & \left[\frac{dw}{dx} V \right] & [0] & [0] \end{bmatrix} dx$$

$$[F^e] = \int_0^h \begin{Bmatrix} \{q(x) w\} \\ \{0\} \\ \{0\} \\ \{0\} \end{Bmatrix} dx,$$

corresponding to nodal values vector:

$$[U^e] = [Mx_1 \quad V_1 \quad \phi_1 \quad w_1 \quad Mx_2 \quad V_2 \quad \phi_2 \quad w_2]^T$$

In short form: $[K^e][U^e] = [F^e]$, where, as it is usually often in mixed theory the highest derivative in either form is first order, consequently C^0 continuity interpolation functions will be used in the approximation for each of the field variables m_x, V, ϕ and w as indicated above, Substitution of these functions into the weak forms yields the following mixed finite element *explicit augmented symmetrical matrix* $[K^e]$ and load vector $[F^e]$ for Timoshenko beam subjected to uniformly distributed load and homogeneous boundary condition:

$$\begin{bmatrix} -\frac{h}{3EI} & 0 & \frac{1}{2} & 0 & -\frac{h}{6EI} & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{h}{3KGA} & -\frac{h}{3} & -\frac{1}{2} & 0 & -\frac{h}{6KGA} & -\frac{h}{6} & \frac{1}{2} \\ \frac{1}{2} & -\frac{h}{3} & 0 & 0 & \frac{1}{2} & -\frac{h}{6} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ -\frac{h}{6EI} & 0 & \frac{1}{2} & 0 & -\frac{h}{3EI} & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{h}{6KGA} & -\frac{h}{6} & -\frac{1}{2} & 0 & -\frac{h}{3KGA} & -\frac{h}{3} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{h}{6} & 0 & 0 & -\frac{1}{2} & -\frac{h}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \end{bmatrix} \& \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{qh}{2} \\ 0 \\ 0 \\ 0 \\ \frac{qh}{2} \end{bmatrix}$$

where for rectangular cross-sectional beam:

$$KGA \equiv K_s GA = \frac{60 EI}{(12 + 11\nu)h^2}, \text{ or } = \frac{5 EI}{(1 + \nu)h^2}$$

according to [12] & [10], respectively.

VI. ILLUSTRATIVE ANALYTICAL EXAMPLES

In order to verify the aforementioned explained revision, a selected single span beam problem of length L , prismatic cross section with constant EI subjected to uniform distributed load UDL; $q(x)$ of intensity q or concentrated load CONL; P at a specified point, will be resolved analytically using both theorems EBBT and TBT in subsequent paragraphs.

A. Determinate Euler-Bernoulli Beams

1) Simply Supported Beam with CONL at Mid-Span

In this case the deflection curve is symmetric about $x = L/2$, thereby it has a *symmetry* condition and then it can be treated as half of the given beam governed by a *homogeneous* differential equation of Euler-Bernoulli Beam. Observing that the symmetric condition now becomes as boundary conditions for half of the beam, then:

$$EI \frac{d^4 w(x)}{dx^4} = 0, 0 \leq x \leq L/2,$$

yields:

$$w^E(x) = \frac{x^3}{6} c_1 + \frac{x^2}{2} c_2 + x c_3 + c_4$$

applying the boundary conditions:

$w(0) = w'(L/2) = -EIw''(0) = 0, -EIw'''(L/2) = P/2$
and in order to obtain dimensionless numerical coefficients for the values of $w^E(x)$ and $M_x(x)$, the required solutions can be rearranged as:

$$w^E(x) = \frac{P L^3}{48 EI} \left[3 \left(\frac{x}{L} \right) - 4 \left(\frac{x}{L} \right)^3 \right]$$

$$\theta_x = \frac{dw^E(x)}{dx} = \frac{P L^2}{16 EI} \left[1 - 4 \left(\frac{x}{L} \right)^2 \right], M_x = \frac{P L}{2} \left(\frac{x}{L} \right)$$

The solution for $L/2 \leq x \leq L$ can be obtained by replacing each x by $(L - x)$ in the last expressions, and the two parts can be combined in terms of unit step (Heaviside) function for complete solution.

2) Simply Supported Beam with UDL

Similarly, but for non-homogenous differential equation:

$$EI \frac{d^4 w(x)}{dx^4} = q(x), 0 \leq x \leq L,$$

yields:

$$w^E(x) = \frac{x^3}{6} c_1 + \frac{x^2}{2} c_2 + x c_3 + c_4 + \frac{qx^4}{24EI}$$

Applying the boundary conditions:

$$w(0) = w(L) = w''(0) = w''(L) = 0$$

and in order to obtain dimensionless numerical coefficients for the values of $w^E(x)$ and $M_x(x)$, the required solutions can be rearranged as:

$$w^E(x) = \frac{q L^4}{24 EI} \left[\left(\frac{x}{L} \right)^4 - 2 \left(\frac{x}{L} \right)^3 + \left(\frac{x}{L} \right) \right]$$

$$M_x = -\frac{q L^2}{2} \left[\left(\frac{x}{L} \right)^2 - \left(\frac{x}{L} \right) \right]$$

3) Left-End Supported Cantilever Beam with CONL

For a beam loaded at its free end and applying the BC's:

$$w(0) = w'(0) = -EIw''(L) = 0, -EIw'''(L) = P,$$

then:

$$EI \frac{d^4 w(x)}{dx^4} = 0,$$

yields:

$$w^E(x) = \frac{P L^3}{12 EI} \left[6 \left(\frac{x}{L} \right)^2 - 2 \left(\frac{x}{L} \right)^3 \right]$$

$$\frac{dw^E(x)}{dx} = \frac{P L^2}{2 EI} \left[2 \left(\frac{x}{L} \right) - \left(\frac{x}{L} \right)^2 \right]$$

$$M_x = PL \left[\left(\frac{x}{L} \right) - 1 \right]$$

4) Left-End Supported Cantilever Beam with UDL

$EI \frac{d^4 w(x)}{dx^4} = q(x)$, and application of BC's:

$$w(0) = w'(0) = -EIw''(L) = 0, -EIw'''(L) = 0$$

yields:

$$w^E(x) = \frac{q L^4}{24 EI} \left[\left(\frac{x}{L} \right)^4 - 4 \left(\frac{x}{L} \right)^3 + 6 \left(\frac{x}{L} \right) \right]$$

$$M_x = -\frac{qL^2}{2} \left[\left(\frac{x}{L} \right)^2 - 2 \left(\frac{x}{L} \right) + 1 \right]$$

B. Indeterminate Euler-Bernoulli Beams

1) Built-in Beam with UDL

Applying BC's: $w(0) = w(L) = w'(0) = w'(L) = 0$, for $EI \frac{d^4 w(x)}{dx^4} = q(x)$, $0 \leq x \leq L$, yields:

$$w^E(x) = \frac{qL^4}{24EI} \left[\left(\frac{x}{L} \right)^4 - 2 \left(\frac{x}{L} \right)^3 + \left(\frac{x}{L} \right)^2 \right]$$

$$M_x = -\frac{qL^2}{2} \left[\left(\frac{x}{L} \right)^2 - \left(\frac{x}{L} \right) + \frac{1}{6} \right]$$

equivalently the last $w^E(x)$ can also be rewritten as:

$$w^E(x) = \frac{qL^4}{24EI} \left[\left(\frac{x}{L} \right)^2 - \left(\frac{x}{L} \right) \right]^2$$

2) Fixed-Hinged Beam with UDL

Applying BC's: $w(0) = w(L) = w'(0) = w''(L) = 0$. Then the required solution is:

$$w^E(x) = \frac{qL^4}{48EI} \left[2 \left(\frac{x}{L} \right)^4 - 5 \left(\frac{x}{L} \right)^3 + 3 \left(\frac{x}{L} \right)^2 \right]$$

$$M_x = -\frac{qL^2}{8} \left[4 \left(\frac{x}{L} \right)^2 - 5 \left(\frac{x}{L} \right) + 1 \right]$$

and

$$\frac{dw^E(x)}{dx} = \frac{qL^3}{48EI} \left[8 \left(\frac{x}{L} \right)^3 - 15 \left(\frac{x}{L} \right)^2 + 6 \left(\frac{x}{L} \right) \right]$$

C. Determinate Timoshenko Beams

1) Simply Supported Beam with central CONL

In similar manner in example: 1- (VI-A), the deflection curve is symmetric about $x = L/2$, then based on the aforementioned *proposed systemized routine*, equations (6) and (7) of Timoshenko Beam and their associated boundary conditions yields:

$$d^3 \phi(x)/dx^3 = 0, \text{ for } 0 \leq x \leq L/2,$$

applying **half beam BC's**: $-EI\phi'(0) = \phi(L/2) = 0$, $-EI\phi''(L/2) = P/2$, & $w(0) = 0$, yields for $0 \leq x \leq L/2$:

$$w^T(x) = \frac{PL^3}{48EI} \left[3 \left(\frac{x}{L} \right) - 4 \left(\frac{x}{L} \right)^3 \right] + \frac{PL}{2K_s GA} \left(\frac{x}{L} \right)$$

which is identical, to what given in example (4.3.1) of Reddy [19], and it is evident that, $w^T(x)$ is composed of two parts, the first one is the deflection due to bending and the other is due to shearing effect which will be discussed in more details in the next example. Then:

$$w^T(x) = w_b + w_s,$$

which are:

$$w_b \equiv w^E(x) = \frac{PL^3}{48EI} \left[3 \left(\frac{x}{L} \right) - 4 \left(\frac{x}{L} \right)^3 \right]$$

$$w_s = \frac{PL}{2K_s GA} \left(\frac{x}{L} \right)$$

Moreover, the first derivative of $w^T(x)$ is given as:

$$\frac{dw^T(x)}{dx} = \frac{PL^2}{16EI} \left[1 - 4 \left(\frac{x}{L} \right)^2 \right] + \frac{P}{2K_s GA}$$

but the bending rotation $\phi(x)$ is:

$$\phi(x) = \frac{PL^2}{16EI} \left[1 - 4 \left(\frac{x}{L} \right)^2 \right]$$

which is in opposite sign to what is given in that Reddy-example, because of utilizing his relation (4.3.7a) in [19], that is; $M(x) = EI \frac{\partial \phi(x)}{\partial x}$ instead of eq.(1) above, i.e., *without the minus sign*, in connection to Reddy expression $\gamma_{xz} = \frac{dw^T(x)}{dx} + \phi(x)$ instead of Timoshenko expression $\gamma_{xz} = \frac{dw^T(x)}{dx} - \phi(x)$, therefore for such reason all obtained bending rotations in the article illustrative examples will be in opposite signs to Reddy solutions if they are available. These last two expressions will be investigated more in fore coming illustrative example: 2- (VI-D). Furthermore, it is evident that the following observation can be made:

- $\frac{d}{dx} w^T(x) - \phi(x) = \frac{P}{2K_s GA} = \frac{V_x}{K_s GA} \equiv \gamma_{xz}$ which in turn implies that $|dw^T(x)/dx| > |\phi(x)|$
- $\phi(x) \neq \frac{d}{dx} w^T(x)$, instead, $\phi(x) = \frac{d}{dx} w^E(x)$
- $\phi(L/2) = 0$, in contrast of $\left. \frac{dw^T(x)}{dx} \right|_{L/2} = \frac{PL}{2K_s GA} \neq 0$

which is a non-usual condition in elementary analysis for such beam case, whereas the section at such position is remaining undistorted as pointed out by Timoshenko [1] page 173.

1) Simply Supported Beam with UDL

Similarly, *systemized routine* for such Timoshenko Beam yields, for eq. (6):

$$\phi(x) = c_1 \frac{x^2}{2} + c_2 x + c_3 + \frac{qx^3}{6EI}$$

applying BC's: $-EI \frac{d\phi(0)}{dx} = 0$, $-EI \frac{d\phi(L)}{dx} = 0$, gives:

$$\phi(x) = \frac{q}{6EI} x^3 - \frac{qL}{4EI} x^2 + c_3$$

Eq. (7): $\frac{d^2 w^T(x)}{dx^2} = \frac{d\phi(x)}{dx} - \frac{q}{K_s GA}$, becomes:

$$\frac{d^2 w^T(x)}{dx^2} = \frac{q}{2EI} (x^2 - Lx) - \frac{q}{K_s GA}$$

$$w^T(x) = \frac{qx^4}{24EI} - \frac{qx^3L}{12EI} - \frac{qx^2}{2K_s GA} + c_1 x + c_2$$

applying BC's: $w(0) = w(L) = 0$, yields as it can be verified by Maple:

$$w^T(x) = \frac{qx^4}{24EI} - \frac{qx^3L}{12EI} - \frac{qx^2}{2K_sGA} + \frac{qL^3x}{24EI} + \frac{qLx}{2K_sGA}$$

Then Eq. (4) becomes:

$$K_sGA(qL^3/24EI) - K_sGA c_3 = 0,$$

in which automatically yields: $c_3 = qL^3/24EI$, therefore:

$$\phi(x) = \frac{q}{6EI}x^3 - \frac{qL}{4EI}x^2 + \frac{qL^3}{24EI}$$

Equivalently with dimensionless numerical coefficients:

$$w^T(x) = \frac{qL^4}{24EI} \left[\left(\frac{x}{L}\right)^4 - 2\left(\frac{x}{L}\right)^3 + \left(\frac{x}{L}\right) \right] + \frac{qL^2}{2K_sGA} \left[\left(\frac{x}{L}\right) - \left(\frac{x}{L}\right)^2 \right]$$

$$\phi(x) = \frac{qL^3}{24EI} \left[4\left(\frac{x}{L}\right)^3 - 6\left(\frac{x}{L}\right)^2 + 1 \right]$$

where the two bending and shear deflections are:

$$w_b \equiv w^E(x) = \frac{qL^4}{24EI} \left[\left(\frac{x}{L}\right)^4 - 2\left(\frac{x}{L}\right)^3 + \left(\frac{x}{L}\right) \right]$$

$$w_s = \frac{qL^2}{2K_sGA} \left[\left(\frac{x}{L}\right) - \left(\frac{x}{L}\right)^2 \right]$$

which are identical, (although K_s is missing print), to what is given in table (4.3.1) by Reddy [19], and also to eq.(2.4.6) given by Wang in [22]; if the typographical mistake x^3/L^3 instead of x^2/L^2 is ignored in w_s -part.

Since $G = E/2(1 + \nu)$ and $I = bt^3/12 \equiv At^2/12$, then $A = 12I/t^2$ and $1/GA = 2(1 + \nu)t^2/12EI$, therefore $w_s(x)$ after multiplying by L^2/L^2 can be written in an alternative form as:

$$w_s = \frac{qL^4}{24EI} \frac{2(1 + \nu)}{K_s} \left(\frac{t}{L}\right)^2 \left[\left(\frac{x}{L}\right) - \left(\frac{x}{L}\right)^2 \right]$$

and the total beam deflection due to **TBT** become as:

$$w^T(x) = \frac{qL^4}{24EI} \left[\left(\frac{x}{L}\right)^4 - 2\left(\frac{x}{L}\right)^3 + \left(\frac{x}{L}\right) \right] + \frac{2(1 + \nu)}{K_s} \left(\frac{t}{L}\right)^2 \left[\left(\frac{x}{L}\right) - \left(\frac{x}{L}\right)^2 \right]$$

In which for some fixed values (except its thickness t) for such beam parameters including $K_s = 5/6$, then as t increases $w^T(x)$ increases as shown in Fig. 5, therefore the deflection $w^T(x)$ as well as total rotation $dw^T(x)/dx$ are depending on (t/L) ratio while the bending rotation $\phi(x)$ as well as bending stresses in this case **remains unaffected**.

Figure 5: Deflection of Simply Supported Timoshenko Beam with UDL for various t/L ratios

2) Left-End Supported Cantilever Beam with CONL

This case is governed by a **homogeneous** differential equation $d^3\phi(x)/dx^3 = 0$, for $0 \leq x \leq L$, then $\phi(x) = c_1x^2/2 + c_2x + c_3$, applying BC's: $\phi(0) = -EI\phi'(L) = 0$, $-EI\phi''(L) = P$ and $w(0) = 0$ yields:

$$w^T(x) = \frac{PL^3}{6EI} \left[3\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)^3 \right] + \frac{PL}{K_sGA} \left(\frac{x}{L}\right)$$

which is identical to table 4.3.1 in Reddy [19], and:

$$\phi(x) = \frac{PL^2}{2EI} \left[2\left(\frac{x}{L}\right) - \left(\frac{x}{L}\right)^2 \right]$$

A noticeable difference between $dw^T(x)/dx$ and $\phi(x)$ is the constant P/K_sGA so that the warpage remaining the same for constant shear as mentioned above, furthermore $dw^T(x)/dx$ for various ratios (t/L) , clarifies the aforementioned non-usual distortion condition at built-in end, although this distortion approaches to $\phi(x) \equiv 0$ for the ratios $t/L < 0.02$, it increases as the ratio increases.

Moreover, if the right-end, instead of the left one is built in, the $-EI\phi''(0) = -P$ due to sign convention for shears, and the solution will be identical to the one obtained by replacing each x by $(L - x)$, in the solution of the left-end case above, that is:

$$w^T(x) = \frac{PL^3}{6EI} \left[\left(\frac{x}{L}\right)^3 - 3\left(\frac{x}{L}\right) + 2 \right] + \frac{PL}{K_sGA} \left[1 - \left(\frac{x}{L}\right) \right]$$

which is also identical to [28], and

$$\phi(x) = \frac{PL^2}{2EI} \left[\left(\frac{x}{L}\right)^2 - 1 \right]$$

3) Left-End Supported Cantilever Beam with UDL

Applying BC's: $\phi(0) = -EI\phi'(L) = 0$, $-EI\phi''(L) = 0$ and $w(0) = 0$, yields:

$$w^T(x) = \frac{qL^4}{24EI} \left[\left(\frac{x}{L}\right)^4 - 4\left(\frac{x}{L}\right)^3 + 6\left(\frac{x}{L}\right)^2 \right] + \frac{qL^2}{2K_sGA} \left[2\left(\frac{x}{L}\right) - \left(\frac{x}{L}\right)^2 \right]$$

which is identical to eq.(2.4.23) in Wang [22], and

$$\phi(x) = \frac{qL^3}{6EI} \left[\left(\frac{x}{L}\right)^3 - 3\left(\frac{x}{L}\right)^2 + 3\left(\frac{x}{L}\right) \right]$$

D. Indeterminate Timoshenko Beams

1) Clamped Supported Beam with UDL

Applying: $\phi(0) = \phi(L) = w(0) = w(L) = 0$, then

$$w(x) = \frac{qL^4}{24EI} \left[\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right) \right]^2 + \frac{qL^2}{2K_sGA} \left[\left(\frac{x}{L}\right) - \left(\frac{x}{L}\right)^2 \right]$$

which is identical to [19], except the misprint of K_s , and

$$\phi(x) = \frac{qL^3}{12EI} \left[2\left(\frac{x}{L}\right)^3 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right) \right]$$

2) Fixed-Hinged Supported Beam with UDL

Eq. (6): $d^3/dx^3 (\phi(x)) = q/EI$, yields:

$$\phi(x) = c_1 \frac{x^2}{2} + c_2 x + c_3 + \frac{qx^3}{6EI}$$

Applying BC's: $\phi(0) = 0, -EI\phi'(L) = 0$, then:

$$\phi(x) = \frac{qL^3}{24EI} \left[4\left(\frac{x}{L}\right)^3 - 6\left(\frac{x}{L}\right)^2 \right] + \frac{1}{2L} [2xL - x^2] C_2$$

Eq. (7): with $w(0) = w(L) = 0$, yields

$$w(x) = \frac{qL^4}{24EI} \left[\left(\frac{x}{L}\right)^4 - 2\left(\frac{x}{L}\right)^3 + \left(\frac{x}{L}\right) \right] + \frac{qL^2}{2K_sGA} \left[\left(\frac{x}{L}\right) - \left(\frac{x}{L}\right)^2 \right] + \frac{L^2}{6} \left[-\left(\frac{x}{L}\right)^3 + 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right) \right] C_2$$

Eq. (4) automatically gives with the aid of Maple:

$$C_2 = \frac{qL^4}{8EI} \frac{K_sGA}{3EI + K_sGA L^2} \equiv \frac{qL^2}{8EI} \frac{K_sGA L^2}{3EI + K_sGA L^2}$$

or $C_2 = (qL^2/8EI)(1/3\alpha + 1)$, where $\alpha = EI/K_sGA L^2$ simplifying $w(x)$, and grouping yields:

$$w(x) = \frac{qL^4}{48EI} \left\{ \left[2\left(\frac{x}{L}\right)^4 - \left(\frac{12\alpha + 5}{3\alpha + 1}\right)\left(\frac{x}{L}\right)^3 + \frac{3}{3\alpha + 1}\left(\frac{x}{L}\right) + \left(\frac{x}{L}\right)\left(\frac{6\alpha}{3\alpha + 1}\right) \right] \right\} + \frac{qL^2}{2K_sGA} \left[\left(\frac{x}{L}\right) - \left(\frac{x}{L}\right)^2 \right]$$

Rebooting of α in the term $\left(\frac{x}{L}\right)\left(\frac{6\alpha}{3\alpha + 1}\right)$ and grouping with $\frac{qL^2}{2K_sGA} \times \frac{4}{4} \left[\left(\frac{x}{L}\right) - \left(\frac{x}{L}\right)^2 \right]$ yields:

$$w(x) = \frac{qL^4}{48EI} \left\{ \left[2\left(\frac{x}{L}\right)^4 - \left(\frac{12\alpha + 5}{3\alpha + 1}\right)\left(\frac{x}{L}\right)^3 + \frac{3}{3\alpha + 1}\left(\frac{x}{L}\right) + \left(\frac{x}{L}\right)\left(\frac{6\alpha}{3\alpha + 1}\right) \right] \right\} + \frac{qL^2}{8K_sGA} \left[\left(\frac{12\alpha + 5}{3\alpha + 1}\right)\left(\frac{x}{L}\right) - 4\left(\frac{x}{L}\right)^2 \right]$$

and similarly:

$$\phi(x) = \frac{qL^3}{48EI} \left[8\left(\frac{x}{L}\right)^3 - 3\left(\frac{12\alpha + 5}{3\alpha + 1}\right)\left(\frac{x}{L}\right)^2 + \left(\frac{6}{3\alpha + 1}\right)\left(\frac{x}{L}\right) \right]$$

Remarks:

- The $w(x)$ -expression is fully identical to the given solution for problem 4.11 entitled **Hinged-Fixed** beam in Reddy [19] page 236, but such entitled Reddy- problem has a different solution structure and the authors believe that the meant in the question is left to right **Fixed-Hinged** beam.
- The bending rotation $\phi(x)$ is structurally identical but in opposite sign as explained above and also depend on shear stiffness K_sGA , provided that the bending moment do the same.
- As discussed in example;1(VI-C) above, the same conclusion can also be drawn for $dw^T(x)/dx - \phi(x)$.

The deflection w^T , rotation ϕ and bending moment M_x of such a problem case are shown in Fig. 6, Fig. 7 & Fig. 8 for various ratios (t/L).

Fig. 6. Deflection curve for Fixed-Hinged Beam.

Fig. 7. Bending rotation for Fixed-Hinged Beam.

Fig. 8. Bending moment for Fixed-Hinged Beam.

Since the obtained $\phi(x)$ in this Timoshenko beam example differs from all foregoing ones because of its dependency on *shear stiffness* coefficient, while its corresponding Euler-Bernoulli beam case does not include such factor, moreover the explicit expression of $\phi(x)$ for such beam-type problem according to Reddy formulation is given in Reddy [19] page 236, then it is possible to compare graphically the following quantities of such beam:

- $dw^T(x)/dx$ obtained in the illustrative example;
- $\phi^T(x)$ obtained in the illustrative example;
- $\phi^T(x)$ from problem 4.11 Reddy-solution [19];
- $dw^E(x)/dx$ of corresponding **EBB**-type.

With the aid of Maple, the graphs of such quantities for (t/L) -ratio = 0.3, as shown in the following Fig.9:

Fig. 9. Comparing of rotations.

It is evident that, from beam left end, passing inflection point of curve concavity up to point of zero shear at which the distorted sections change their sliding directions then from this point up to beam right end, the following statement is hold: $|dw^T/dx| > |dw^E/dx| \geq |\phi(x)|$, which reflect the convincing logicity of the utilized Timoshenko expression:

$$dw^T(x)/dx = (\phi + \varphi),$$

i.e.,

$$\gamma_{xz} = dw^T(x)/dx - \phi(x).$$

3) Built-in Beam with UDL & Central Point Load

This problem although it can be solved by superposition technique, it will be treated in similar manner of earlier simply supported beam with concentrated load, then the half beam governed by the **nonhomogeneous** differential equation for UDL only:

$$d^3\phi(x)/dx^3 = q, \text{ for } 0 \leq x \leq L/2$$

then:

$$\phi(x) = c_1 \frac{x^2}{2} + c_2 x + c_3 + \frac{qx^3}{6EI}$$

applying half beam boundary and beam symmetric conditions: $\phi(0) = \phi(L/2) = 0$, $-EI\phi''(L/2) = P/2$, in addition to $w(0) = 0$. Then for $0 \leq x \leq L/2$:

$$\phi(x) = \frac{qL^3}{12EI} \left[2 \left(\frac{x}{L} \right)^3 - 3 \left(\frac{x}{L} \right)^2 + \left(\frac{x}{L} \right) \right] + \frac{PL^2}{8EI} \left[\left(\frac{x}{L} \right) - 2 \left(\frac{x}{L} \right)^2 \right]$$

In which its first part is in opposite sign to what is given by Reddy in his example 4.3.2 by eq. 4.3.25 in Ref. [19], as it is must be due to reason explained in example 1(VI-C) but the second one doesn't do that although its similar to the case in example 3(VI-C) mentioned above, when P & L replaced by P/2 & L/2 respectively, and accordingly the second part of Reddy solution should be also with minus sign. Therefore, in the absence of the reason to the authors for this contradicting case, obviously; in general, the typographical mistakes in foundational references are the major confusing difficulties encounters the interested *beginners* in advanced topics. The deflection $w^T(x)$ in compact form is given by:

$$w^T(x) = \frac{qL^4}{24EI} \left[\left(\frac{x}{L} \right)^2 - \left(\frac{x}{L} \right) \right]^2 + \frac{qL^2}{2K_sGA} \left[\left(\frac{x}{L} \right) - \left(\frac{x}{L} \right)^2 \right] + \frac{PL^3}{48EI} \left[3 \left(\frac{x}{L} \right)^2 - 4 \left(\frac{x}{L} \right)^3 \right] + \frac{PL}{2K_sGA} \left(\frac{x}{L} \right)$$

which fortunately intact from such problems and is fully identical to the one given by indicated Reddy solution.

As a closure to these detailed discussed illustrative examples; the established relationships between deflection, rotation, bending moment, and shear-force of Timoshenko beam theory (**TBT**) and their corresponding quantities due to Euler-Bernoulli beam theory (**EBBT**) given by Wang [22] in the Table 2.2.1, can be also used *if desired* in connection with the above obtained solutions of illustrative examples but only their term $(-dw^E/dx)$ should be replaced by $(+dw^E/dx)$, because of his utilizing $M_x = EI \partial/\partial x \phi(x)$ and $\gamma_{xz} = dw^T(x)/dx + \phi(x)$ instead of what have been utilized in the above examples.

VII. MIXED FINITE ELEMENT SOLUTIONS

To this end, it is quite evident that more complicated applied loading patterns to Timoshenko beam resulted in very complicated analytical solutions even through superposition technique. Therefore, the numerical solutions might be the better choice to tackle such problems, which for example can

be carried out using the TBT explicit augmented matrices developed above which in turn can be implemented in any computer programming language (*authors develop this implementation in a visual basic VB6 programing language to obtain the solutions of the foregoing illustrative examples as shown in next tables*)

Therefore, for fixed given beam parameters including $K_s = 5/6, v = 3/10$ & $t/L = 0.3$, the **Mixed Finite Element Method (MFEM)** utilizing 40-mesh elements yields the results for the deflection w and bending moment m_x based on **TBT**, gives good agreement results as shown in tables [2]-[8], coupled with theoretical results based on **EBT** & **TBT** derived above, for comparing purposes.

TABLE I: SIMPLY SUPPORTED BEAM WITH MIDDLE COL

x	w			m_x		
	EBT	TBT MFEM		EBT	TBT	MFEM
0.0*L	0.00000	0.00000	0.00000	0.00	0.00	0.0000
0.1*L	0.00617	0.00734	0.00734	0.05	0.05	0.0500
0.2*L	0.01183	0.01417	0.01417	0.10	0.10	0.1000
0.3*L	0.01650	0.02001	0.02001	0.15	0.15	0.1499
0.4*L	0.01967	0.02435	0.02435	0.20	0.20	0.1999
0.5*L	0.02083	0.02668	0.02668	0.25	0.25	0.2499
0.6*L	0.01967	0.02435	0.02435	0.20	0.20	0.1999
0.7*L	0.01650	0.02001	0.02001	0.15	0.15	0.1499
0.8*L	0.01183	0.01417	0.01417	0.10	0.10	0.1000
0.9*L	0.00617	0.00734	0.00734	0.05	0.05	0.0500
1.0*L	0.00000	0.00000	0.00000	0.00	0.00	0.0000
factor	PL ³ /EI			PL		

TABLE II: SIMPLY SUPPORTED BEAM WITH UDL

x	w			m_x		
	EBT	TBT MFEM		EBT	TBT	MFEM
0.0*L	0.00000	0.00000	0.00000	0.0000	0.0000	0.0000
0.1*L	0.00409	0.00514	0.00514	0.0450	0.0450	0.0450
0.2*L	0.00773	0.00960	0.00961	0.0800	0.0800	0.0800
0.3*L	0.01059	0.01304	0.01305	0.1050	0.1050	0.1050
0.4*L	0.01240	0.01521	0.01521	0.1200	0.1200	0.1200
0.5*L	0.01302	0.01595	0.01595	0.1250	0.1250	0.1250
0.6*L	0.01240	0.01521	0.01521	0.1200	0.1200	0.1200
0.7*L	0.01059	0.01304	0.01305	0.1050	0.1050	0.1050
0.8*L	0.00773	0.00960	0.00961	0.0800	0.0800	0.0800
0.9*L	0.00409	0.00514	0.00514	0.0450	0.0450	0.0450
1.0*L	0.00000	0.00000	0.00000	0.0000	0.0000	0.0000
factor	qL ⁴ /EI			qL ²		

TABLE III: LEFT-END SUPPORTED CANTILEVER BEAM WITH CONL

x	w			m_x		
	EBT	TBT MFEM		EBT	TBT	MFEM
0.0*L	0.00000	0.00000	0.00000	-1.00	-1.00	-0.9990
0.1*L	0.00483	0.00717	0.00718	-0.90	-0.90	-0.8990
0.2*L	0.01867	0.02335	0.02335	-0.80	-0.80	-0.7991
0.3*L	0.04050	0.04752	0.04753	-0.70	-0.70	-0.6992
0.4*L	0.06933	0.07869	0.07870	-0.60	-0.60	-0.5993
0.5*L	0.10467	0.11587	0.11587	-0.50	-0.50	-0.4994
0.6*L	0.14400	0.15804	0.15805	-0.40	-0.40	-0.3997
0.7*L	0.18783	0.20421	0.20422	-0.30	-0.30	-0.2998
0.8*L	0.23467	0.25339	0.25339	-0.20	-0.20	-0.1998
0.9*L	0.28350	0.30456	0.30456	-0.10	-0.10	-0.0999
1.0*L	0.33333	0.35673	0.35673	0.00	0.00	0.0000
factor	PL ³ /EI			PL		

TABLE IV: LEFT-END SUPPORTED CANTILEVER BEAM WITH UDL

x	w			m_x		
	EBT	TBT MFEM		EBT	TBT	MFEM
0.0*L	0.00000	0.00000	0.00000	-0.500	-0.500	-0.500
0.1*L	0.00234	0.00456	0.00457	-0.405	-0.405	-0.405
0.2*L	0.00873	0.01294	0.01296	-0.320	-0.320	-0.320
0.3*L	0.01834	0.02430	0.02432	-0.245	-0.245	-0.245
0.4*L	0.03040	0.03789	0.03791	-0.180	-0.180	-0.180
0.5*L	0.04427	0.05304	0.05307	-0.125	-0.125	-0.125
0.6*L	0.05940	0.06923	0.06926	-0.080	-0.080	-0.080
0.7*L	0.07534	0.08598	0.08602	-0.045	-0.045	-0.045
0.8*L	0.09173	0.10296	0.10301	-0.020	-0.020	-0.020
0.9*L	0.10834	0.11992	0.11996	-0.005	-0.005	-0.005
1.0*L	0.12500	0.13670	0.13675	0.000	0.000	0.000
factor	qL ⁴ /EI			qL ²		

TABLE V: CLAMPED SUPPORTED BEAM WITH UDL

x	w			m_x		
	EBT	TBT MFEM		EBT	TBT	MFEM
0.0*L	0.0000	0.00000	0.00000	-0.0833	-0.0833	-0.0831
0.1*L	0.0003	0.00139	0.00139	-0.0383	-0.0383	-0.0381
0.2*L	0.0011	0.00294	0.00294	-0.0033	-0.0033	-0.0031
0.3*L	0.0018	0.00429	0.00430	0.0217	0.0217	0.0219
0.4*L	0.0024	0.00521	0.00522	0.0367	0.0367	0.0369
0.5*L	0.0026	0.00553	0.00554	0.0417	0.0417	0.0419
0.6*L	0.0024	0.00521	0.00522	0.0367	0.0367	0.0369
0.7*L	0.0018	0.00429	0.00430	0.0217	0.0217	0.0219
0.8*L	0.0011	0.00294	0.00294	-0.0033	-0.0033	-0.0031
0.9*L	0.0003	0.00139	0.00139	-0.0383	-0.0383	-0.0381
1.0*L	0.0000	0.00000	0.00000	-0.0833	-0.0833	-0.0831
factor	qL ⁴ /EI			qL ²		

TABLE VI: FIXED-HINGED BEAM WITH UDL

x	w			m_x		
	EBT	TBT MFEM		EBT	TBT	MFEM
0.0*L	0.00000	0.000000	0.00000	-0.1250	-0.1168	-0.1165
0.1*L	0.00053	0.00181	0.00181	-0.0675	-0.0601	-0.0598
0.2*L	0.00173	0.00400	0.00400	-0.0200	-0.0134	-0.0132
0.3*L	0.00315	0.00610	0.00610	0.0175	0.0232	0.0235
0.4*L	0.00440	0.00773	0.00774	0.0450	0.0499	0.0501
0.5*L	0.00521	0.00865	0.00865	0.0625	0.0666	0.0668
0.6*L	0.00540	0.00867	0.00867	0.0700	0.0733	0.0734
0.7*L	0.00490	0.00773	0.00774	0.0675	0.0699	0.0701
0.8*L	0.00373	0.00587	0.00587	0.0550	0.0566	0.0567
0.9*L	0.00203	0.00321	0.00322	0.0325	0.0333	0.0334
1.0*L	0.00000	0.00000	0.00000	0.0000	0.0000	0.0000
factor	qL ⁴ /EI			qL ²		

TABLE VII: BUILT-IN BEAM WITH UDL & CENTRAL POINT LOAD

x	w			m_x		
	EBT	TBT MFEM		EBT	TBT	MFEM
0.0*L	0.00000	0.00000	0.00000	-0.2083	-0.2083	-0.2082
0.1*L	0.00088	0.00310	0.00310	-0.1133	-0.1133	-0.1132
0.2*L	0.00290	0.00711	0.00711	-0.0283	-0.0283	-0.0283
0.3*L	0.00521	0.01118	0.01118	0.0467	0.0467	0.0467
0.4*L	0.00707	0.01455	0.01455	0.1117	0.1117	0.1116
0.5*L	0.00781	0.01659	0.01659	0.1667	0.1667	0.1666
0.6*L	0.00707	0.01455	0.01455	0.1117	0.1117	0.1116
0.7*L	0.00521	0.01118	0.01118	0.0467	0.0467	0.0467
0.8*L	0.00290	0.00711	0.00711	-0.0283	-0.0283	-0.0283
0.9*L	0.00088	0.00310	0.00310	-0.1133	-0.1133	-0.1132
1.0*L	0.00000	0.00000	0.00000	-0.2083	-0.2083	-0.2082

all previous results are due: $q = P = L = EI = 1, v = 0.3$ & $t/L = 0.3$

VIII. DISCUSSION AND CONCLUSION

From preceding illustrative examples results and their discussions, the following can be easily concluded:

- Evaluations of $w(x)$ shows that the effect of shear deformation is to increase the deflection.
- The contribution due to shear deformation to the deflection depends on the modulus ratio E/G as well as the ratio of thickness to length t/L .
- The effect of shear deformation is negligible for thin and long beams whereas it is more significant for beams with thickness-to-length ratios $> 1/10$.
- The bending rotation $\phi(x)$ as well as bending moment and axial bending stresses are independent of shear deformation for all statically determinate and indeterminate beams if they obey symmetric boundary condition and loading, whereas for general statically indeterminate beams they will be affected by shear stiffness K_sGA .
- The described total rotation by Timoshenko is evidentiary apparent and interpreted.
- The superiority of results due mixed formulation is evident in all field variables of the beam problem.

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DEDICATION

This Article is Dedicated to the Memory of Pioneer and Mentor; My Teacher Professor Salah Yahya AL-Barony. (1944 - 9Aug.2020)

First Author

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