Static and Buckling Analysis of a Three-Dimensional (3-D) Rectangular Thick Plates Using Exact Polynomial Displacement Function

F. C. Onyeka, T. E. Okeke, and C. D. Nwa-David

Abstract — **This paper is devoted to study the buckling response of axially compressed rectangular thick plate based on the exact polynomial potential functional. The governing and equilibrium equation of an isotropic plate was derived based on the three-dimensional (3-D) static theory of elasticity, to get the relations between the rotations and deflection. These equations are solved in the form of polynomial analytically to obtain the exact displacements and stresses that are induced due to uniaxial compressive load action on the plate. By incorporating deflection and rotation function into the fundamental equation and minimized with respect to deflection coefficient, a new expression of the determination of the critical buckling load was established. This expression was applied to solve the buckling problem of a clamped thick rectangular plate which was simply supported at the first and freely supported at the third edge (SCFC). A graphic representation of results showed that, as the aspect ratio of the plate increases, the value of critical buckling load decreases while as critical buckling load increases as the length to breadth ratio increases. This implies that an increase in plate width increases the chance of failure in a plate structure. This theory obviates the numerical approximations in the thickness direction thereby guaranteed accuracy in the solution of the displacement along the direction of thickness axis of the plate, hence, a significant lessening of the cost of computation.**

Key words — **Exact displacement and stress, buckling load, three-dimensional (3-D) static theory of elasticity, SCFC thick rectangular plate.**

I. INTRODUCTION

Basic three-dimensional structural elements with plane and straight surfaces whose thickness is geometrically small compared to other dimensions known as plates [1]; Have an increased research interest among engineers and scientists as they are widely applied in the construction of ships, railways, bridges, aerospace, mechanical and structural engineering [2], [3] due to their cost benefits, high strength, lightweight and load resistance properties [4].

With respect to varying applications, plates could be laminated, homogeneous or functionally graded materials with different sizes, thicknesses, and shapes. Considering edge conditions, plates can be clamped, simply supported, and of free boundary conditions [5], based on shape as: elliptical, triangular, skew plates, square, circular, and rectangular, also based on material composition as: anisotropic and isotropic plates. Depending on depth, plates have been classified as thin, thick, or moderately thick [6]-

[8]. Furthermore, [9]-[11] has defined rectangular plates with $50 \leq a/t \leq 100$ as thin plate, $20 \leq a/t \leq 50$ as moderately thick and $a/t \leq 20$ as thick plate where a/t is the span-to-depth ratio.

The significance and large application of thick plates has attracted more researchers to investigate its potency for most engineering applications. Areas of investigation include bending, vibration and buckling [12]. Thicker plates are frequently exposed to normal, shearing, and compressive loads acting in the center plane of the plate. Plate buckling can occur under certain load conditions. Buckling is a phenomenon where at a critical compressive load value, a material under the influence of in-plane compressive loads moves stable to unstable state of equilibrium. To avoid premature catastrophic structural failures, determining the buckling abilities of plates is of considerable practical significance.

Classical plate theory (CPT), first order shear deformation plate theory (FSDT) [10], and the higher order shear deformation plate theories (HSDT) [11], have been developed and applied by several researchers to solve plate problems such as buckling. However, the Kirchhoff classical plate theory [13] gives accurate results for the buckling of thin plates only as a result of the neglected transverse shear deformation. In thick plates, the transverse shear deformation is strongly marked. Mindlin [14] considered the effect of transverse shear deformation in the first order shear deformation theory that is displacement based but shear correction factor was required. In order to compensate for the deficiencies of the classical plate theory and first order shear deformation theory, higher order shear deformation theories (HSDTs) were developed. Without the shear correction factor, the HSDTs provides a shear stress condition that is zero at the upper and bottom part of the plate. Although HSDTs are suitable for thick plate analysis, they can't give accurate result of typical three-dimensional plate analysis. To obtain an exact solution for a three-dimensional plate analysis, 3D theory is required; hence this research work is needful.

To study the buckling behavior of thick plates, the equilibrium, numerical, or energy methods can be employed. Equilibrium method, also known as Euler method, adds up all the loads acting on a continuum to zero and obtains the governing differential equation solution by integrating directly and fulfilling the boundary conditions of the four edges of the plate. The outcome of numerical methods is

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F. C. Onyeka, Edo University Iyamho, Edo State, Nigeria.
(e-mail: onyeka.festus@edouniversity.edu.ng)
T. E. Okeke, University of Nigeria, Nsukka, Nigeria.
(e-mail: edozie.okeke@unn.edu.ng)

C. D. Nwa-David, Michael Okpara University of Agriculture, Umaudike, Nigeria.
(e-mail: nwadavid.chidobere@mouau.edu.ng)

approximate numerical solutions of the plate problem. This method consists of boundary element methods, finite strip methods, truncated double Fourier series and finite difference methods [15]. This method consumes more time and numerous works have to be done in order to get an exact solution.

Energy method is completely different from equilibrium and numerical approaches as it sums all the strain energy and potential energy or external work on the continuum to be same with total potential energy [16]. To obtain the stability matrix, the total potential energy function is minimized. In this study, a unified approach (equilibrium-energy) is employed.

Onyeka *et al.* [17] applied variational energy method to analyze the buckling of a thick CSSS plate under uniaxial compressive load with a new trigonometric shear deformation plate theory. For the formulation of the energy equation, there was no need for shear correction factors and unlike refined plate theories, the new theory developed for the analysis considered all the stress elements of the plate. However, the authors did not employ polynomial displacement function and SCFC boundary condition was not considered.

A refined trigonometric shear deformation theory was developed by Gunjal *et al.* [18] to investigate the buckling of simply supported isotropic rectangular plate under uniaxial and biaxial compression. The virtual work principle was employed to obtain the governing equations and boundary conditions of the theory. Neglecting the use of shear correction factor, the shear stresses free conditions at the plate surfaces were satisfied by the theory. The authors did not consider a typical 3D plate theory for their analysis and a thick plate with the SCFC boundary condition was not taken into account.

Both polynomial, and trigonometric functions with an energy method was employed by Onyeka *et al.* [19] to obtain the buckling solution on a thick plate with all edges clamped. The authors formulated the total potential energy equation from 3D constitutive relations and to obtain the relations between the deflections and shear deformation rotation along the direction of x and y coordinates, the compatibility equations were established. The authors did not consider plates with SCFC edge condition and equilibrium-energy approach as in this present study.

Applying the alternative II theory based on polynomial shape function, Ibearugbulem *et al.* [20] evaluated the stability of thick plates using energy approach. The authors combined the strain energy and external work to obtain the total potential energy which was reduced to the governing equations. To get the critical buckling load, the authors substituted the polynomial function into the governing equation and the results obtained was compared with those of first order shear deformation theory. However, the theory considered by the authors is an incomplete 3D approach because it neglects all the stress and strain along the thickness direction of the plate and this negligence do not allow for the full estimation of the plate buckling load. Also plates with the SCFC support condition was not addressed.

Sayyad and Ghugal [21] employed shear exponential deformation theory to analyze the buckling of thick plates subjected to biaxial and uniaxial in-plane forces. The polynomial displacement function was not taken into

consideration. Ibeabuchi *et al.* [22] used work principal approach to investigate the buckling of uniaxially compressed plate elastically restrained in all directions. With polynomial displacement function, the authors obtained the buckling coefficients of the plate and developed a numerical model. The assumption made in their study is limited to classical plate theory. Onah *et al.* [23] applied single Fourier sine transformation approach to obtain the solution for Kirchhoff CCSS rectangular plate buckling. Their study can only predict buckling load of thin or moderately thick plates because there was consideration for the plate's thickness. The study in [21]-[23] did not cover thick plates with the SCFC boundary condition or used an exact three-dimensional plate theory in their analysis.

With displacement potential function approach, Vareki *et al.* [24], Uymaz *et al.* [25] Singh *et al.* [26] and Lee [27] obtained the buckling solution of simply supported thick plates. The 3D plate theory was not employed, and the displacement function applied were not a derivative of the compatibility equation. Both authors did not incorporate the equilibrium approach in their energy method and SCFC plate's boundary condition were not also considered.

With an exact solution approach, Onyeka *et al.* [28] analyzed the stability of a thick plate that is simply supported at the first and fourth edges, clamped and freely supported in the second and third edge respectively (SCFS). The authors' exact formulation of the potential energy equation was obtained from the compatibility equation to get a close form solution of the polynomial and trigonometric displacement functions. Also, the study carried out by Onyeka *et al.* [29] to investigate the stability of all edges simply supported (SSSS) thick plates confirmed that exact 3-D plate theory using polynomial and trigonometric displacement function gives a good solution. Although a typical 3-D thick plate was covered in their analysis, they did not address a thick plate that is simply supported and free at the first, third edge and clamped at the second and fourth edge (SCFC).

From the previous studies reviewed, it is found that the boundary condition of the plate in this present study has not been considered for buckling analysis, and not much has been done with 3-D plate theory, which consists of all the six strains and stress components that is required for a typical thick plate analysis. This gap in the literature is worth filling. This study is aimed at bridging the gap in literature by developing a new polynomial displacement function and applying it in the exact three-dimensional stability analysis of isotropic thick rectangular plate subjected to an in-plane loading. The main objective of this study is to formulate a realistic formula for calculating the critical buckling load of thick rectangular plate simply supported at one edge, free at one edge and clamped at the two outer edges (SCFC) under uniaxial compressive load, using fourth order polynomial displacement function and equilibrium-energy approach.

II. METHODOLOGY

A. Potential Energy Equation Formulation

The energy equation for an axially loaded rectangular thick plate is formulated by considering a thick plate assumption, with the x-z section and y-z section, which are initially normal to the x-y plane before bending off the normal to the

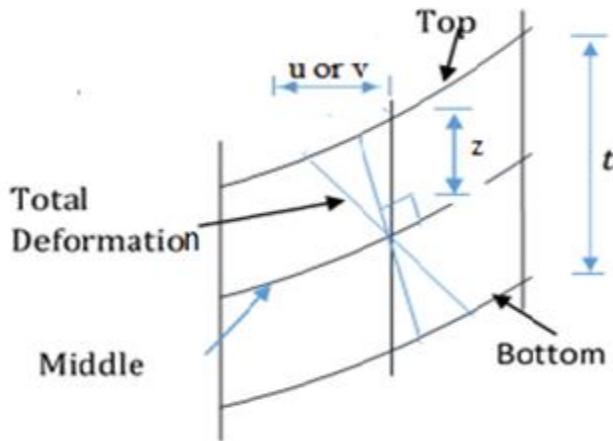x-y plane after bending of the plate as shown in the section of plate presented in the Fig. 1.

Fig 1. The displacement of section x-z or y-z of the plate.

The non-dimensional total potential energy [Π] expression for an elastic three-dimensional plate theory of R and Q coordinates at the span-thickness aspect ratio (a/t) is in line with [17] and presented as:

$$
\begin{aligned}
\Pi &= D^* \frac{(1-\mu)ab}{2a^2(1-2\mu)} \int_0^1 \int_0^1 \Bigg[(1-\mu)\left(\frac{\partial\theta_{sx}}{\partial R}\right)^2 + \frac{1}{\beta}\frac{\partial\theta_{sx}}{\partial R}\cdot\frac{\partial\theta_{sy}}{\partial Q} \\
&+ \frac{(1-\mu)}{\beta^2}\left(\frac{\partial\theta_{sy}}{\partial Q}\right)^2 + \frac{(1-2\mu)}{2\beta^2}\left(\frac{\partial\theta_{sx}}{\partial Q}\right)^2 \\
&+ \frac{(1-2\mu)}{2}\left(\frac{\partial\theta_{sy}}{\partial R}\right)^2 \\
&+ \frac{6(1-2\mu)}{t^2}\left(a^2\theta_{sx}{}^2 + a^2\theta_{sy}{}^2 + \left(\frac{\partial w}{\partial R}\right)^2 + \frac{1}{\beta^2}\left(\frac{\partial w}{\partial Q}\right)^2\right. \\
&+ \left. 2a.\theta_{sx}\frac{\partial w}{\partial R} + \frac{2a.\theta_{sy}}{\beta}\frac{\partial w}{\partial Q}\right) + \frac{(1-\mu)a^2}{t^4}\left(\frac{\partial w}{\partial S}\right)^2 \\
&- \frac{N_x}{D^*}\cdot\left(\frac{\partial w}{\partial R}\right)^2 \Bigg] \partial R \partial Q
\end{aligned}
\tag{1}
$$

given that D^* is the Rigidity for 3-D thick plate, let

$$
D^* = D\,\frac{(1-\mu)}{(1-2\mu)}
$$

where D is the Rigidity of the CPT or incomplete 3-D thick plate, let $N_x, \mu, w, \theta_{Sx},$ and θ_{Sy} are the uniform applied uniaxial compression load of the plate, the poison ratio, deflection , shear deformation rotation along x axis and shear deformation rotation along y axis respectively.

B. Compatibility Equation

The true compatibility equations in x-z plane and y-z plane according to [19] is obtained by minimizing the energy equation with respect to rotation in x-z plane and rotation in y-z plane and equate its integrands to zero to get:

$$
(1-\mu)\frac{\partial^2\theta_{sx}}{\partial R^2} + \frac{1}{2\beta}\cdot\frac{\partial^2\theta_{sy}}{\partial R\partial Q} + \frac{(1-2\mu)}{2\beta^2}\frac{\partial^2\theta_{sx}}{\partial Q^2}
$$
$$
+ \frac{6(1-2\mu)}{t^2}\left(a^2\theta_{sx} + a.\frac{\partial w}{\partial R}\right) = 0
\tag{2}
$$

$$
\frac{1}{2\beta}\cdot\frac{\partial^2\theta_{sx}}{\partial R\partial Q} + \frac{(1-\mu)}{\beta^2}\frac{\partial^2\theta_{sy}}{\partial Q^2} + \frac{(1-2\mu)}{2}\frac{\partial^2\theta_{sy}}{\partial R^2}
$$
$$
+ \frac{6(1-2\mu)}{t^2}\left(a^2\theta_{sy} + \frac{a.}{\beta}\frac{\partial w}{\partial Q}\right) = 0
\tag{3}
$$

Using the law of addition, (2) and (3) will be simplified, then factorizing the outcome gives:

$$
\frac{\partial w}{\partial R}\Bigg[(1-\mu)\frac{\partial^2}{\partial R^2} + \frac{1}{\beta^2}\cdot\frac{\partial^2}{\partial Q^2}(1-\mu) + \frac{6(1-2\mu)a^2}{t^2}\cdot\left(1+\frac{1}{c}\right)\Bigg] = 0
\tag{4}
$$

$$
\frac{1}{\beta}\cdot\frac{\partial w}{\partial Q}\Bigg[\frac{\partial^2}{\partial R^2}(1-\mu) + \frac{(1-\mu)}{\beta^2}\frac{\partial^2}{\partial Q^2} + \frac{6(1-2\mu)a^2}{t^2}\cdot\left(1+\frac{1}{c}\right)\Bigg] = 0
\tag{5}
$$

After simplification using law of addition, one of the possible of equation becomes:

$$
\frac{6(1-2\mu)(1+c)}{t^2} = -\frac{c(1-\mu)}{a^2}\left(\frac{\partial^2}{\partial R^2} + \frac{1}{\beta^2}\frac{\partial^2}{\partial Q^2}\right)
\tag{6}
$$

C. General Governing Equations

The minimization of energy equation with respect to deflection gives the general governing equation as presented in [29]:

$$
\begin{aligned}
\frac{D^*}{2a^2}\int_0^1\int_0^1 &\Bigg[\frac{6(1-2\mu)(1+c)}{t^2}\left(\frac{\partial^2 w}{\partial R^2} + \frac{1}{\beta^2}\cdot\frac{\partial^2 w}{\partial Q^2}\right) \\
&+ \frac{(1-\mu)a^2}{t^4}\frac{\partial^2 w}{\partial S^2} - \frac{N_x}{D^*}\cdot\frac{\partial^2 w}{\partial R^2}\Bigg]dR\,dQ \\
&= 0
\end{aligned}
\tag{7}
$$

Substituting (6) into (7) and simplifying the outcome gives two governing differential equations of a 3-D rectangular plate subject to pure buckling as presented in (8) and (9):

$$
\frac{\partial^4 w_1}{\partial R^4} + \frac{2}{\beta^2}\cdot\frac{\partial^4 w_1}{\partial R^2\partial Q^2} + \frac{1}{\beta^4}\cdot\frac{\partial^4 w_1}{\partial Q^4} - \frac{N_{x1}a^4}{gD^*}\cdot\frac{\partial^2 w_1}{\partial R^2} =
\tag{8}
$$

$$
\frac{(1-\mu)a^4}{t^4}\cdot\frac{\partial^2 w_S}{\partial S^2} - \frac{N_{xs}a^4}{D^*}\cdot\frac{\partial^2 w_S}{\partial R^2} = 0
\tag{9}
$$

Thus, the polynomial expression for deflection derived from (8) according to Onyeka *et al.* [30] is presented in (10) as:

$$
w = \Delta_0\,(a_0 + a_1 R + a_2 R^2 + a_3 R^3 + a_4 R^4) \times (b_0 + b_1 Q + b_2 Q^2 + b_3 Q^3 + b_4 Q^4)
\tag{10}
$$

Equation (10) can be re-written in line with the work of Onyeka *et al.* [30] as:

$$
w = A_1 h
\tag{11}
$$

where:

$$A_1 = \Delta_0 \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} . \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \tag{12}$$

$$h = [1 \ R \ R^2 \ R^3 \ R^4].[1 \ Q \ Q^2 \ Q^3 \ Q^4] \tag{13}$$

$$\theta_{sx} = \frac{A_2}{a} . \frac{\partial h}{\partial R} \tag{14}$$

$$\theta_{sy} = \frac{A_3}{a\beta} . \frac{\partial h}{\partial Q} \tag{15}$$

Given that: h is the shape function of the plate, A_1 is the coefficient of deflection A_2 and A_3 are the coefficients of shear deformation in x axis and y axis respectively.

D. Direct Governing Equations

By substituting (11), (14) and (15) into (1), the Energy equation becomes:

$$
\begin{aligned}
\Pi &= \frac{D^* ab}{2a^4} \Bigg[(1-\mu)A_2{}^2 \int_0^1 \int_0^1 \left(\frac{\partial^2 h}{\partial R^2}\right)^2 dRdQ \\
&+ \frac{1}{\beta^2} \Bigg[A_2.A_3 + \frac{(1-2\mu)A_2{}^2}{2} \\
&+ \frac{(1-2\mu)A_3{}^2}{2} \Bigg] \int_0^1 \int_0^1 \left(\frac{\partial^2 h}{\partial R\partial Q}\right)^2 \\
&+ \frac{(1-\mu)A_3{}^2}{\beta^4} \int_0^1 \int_0^1 \left(\frac{\partial^2 h}{\partial Q^2}\right)^2 dRdQ \\
&+ 6(1 \\
&- 2\mu)\left(\frac{a}{t}\right)^2 \Bigg([A_2{}^2 + A_1{}^2 + 2A_1A_2]. \int_0^1 \int_0^1 \left(\frac{\partial h}{\partial R}\right)^2 dRdQ \\
&+ \frac{1}{\beta^2}.[A_3{}^2 + A_1{}^2 + 2A_1A_3]. \int_0^1 \int_0^1 \left(\frac{\partial h}{\partial Q}\right)^2 dRdQ \Bigg) \\
&- \frac{N_x a^2 A_1{}^2}{D^*} . \int_0^1 \int_0^1 \left(\frac{\partial h}{\partial R}\right)^2 dRdQ \Bigg]
\end{aligned}
\tag{16}
$$

Differentiating (17) with respect to shear deformation coefficient (A_2 and A_3), and solve simultaneously gives:

$$A_2 = \left(\frac{k_{12}k_{23} - k_{13}k_{22}}{k_{12}k_{12} - k_{11}k_{22}}\right).A_1 \tag{17}$$

$$A_3 = \left(\frac{k_{12}k_{13} - k_{11}k_{23}}{k_{12}k_{12} - k_{11}k_{22}}\right).A_1 \tag{18}$$

Let:

$$
\begin{aligned}
k_{11} &= (1-\mu)k_{RR} + \frac{1}{2\beta^2}(1-2\mu)k_{RQ} \\
&+ 6(1-2\mu)\left(\frac{a}{t}\right)^2 k_R
\end{aligned}
\tag{19}
$$

$$k_{21} = k_{12} = \frac{1}{2\beta^2} k_{RQ}; \quad k_{13} = -6(1-2\mu)\left(\frac{a}{t}\right)^2 k_R \tag{19b}$$

$$k_{32} = k_{23} = -\frac{6}{\beta^2}(1-2\mu)\left(\frac{a}{t}\right)^2 k_Q \tag{20}$$

$$
\begin{aligned}
k_{22} &= \frac{(1-\mu)}{\beta^4} k_{QQ} + \frac{1}{2\beta^2}(1-2\mu)k_{RQ} \\
&+ \frac{6}{\beta^2}(1-2\mu)\left(\frac{a}{t}\right)^2 k_Q
\end{aligned}
\tag{21}
$$

Differentiating (16) with respect to deflection coefficient (A_1) and simplifying the outcome, an expression for the critical buckling load (N_{xcr}) is established as:

$$
\frac{N_x a^2}{\pi^2 D} = 6(1-2\mu)\left(\frac{a}{t}\right)^2 \left(\begin{array}{c} \left[1 + \left(\dfrac{k_{12}k_{23} - k_{13}k_{22}}{k_{12}k_{12} - k_{11}k_{22}}\right)\right] \\ + \dfrac{1}{\beta^2}.\left[1 + \left(\dfrac{k_{12}k_{13} - k_{11}k_{23}}{k_{12}k_{12} - k_{11}k_{22}}\right)\right].\dfrac{k_Q}{k_R} \end{array} \right)
\tag{22}
$$

Similarly:

$$
N_x = \frac{6\pi^2(1-2\mu)D}{a^2}\left(\frac{a}{t}\right)^2 \left(\begin{array}{c} \left[1 + \left(\dfrac{k_{12}k_{23} - k_{13}k_{22}}{k_{12}k_{12} - k_{11}k_{22}}\right)\right] \\ + \dfrac{1}{\beta^2}.\left[1 + \left(\dfrac{k_{12}k_{13} - k_{11}k_{23}}{k_{12}k_{12} - k_{11}k_{22}}\right)\right].\dfrac{k_Q}{k_R} \end{array} \right)
\tag{23}
$$

where
D is the modulus of Rigidity and β represents the ratio of length and breadth of the plate.

E. Numerical Analysis

A buckling problem of a clamped thick rectangular plate which was simply supported at the first and freely supported at the third edge (SCFC) and subjected to a uniaxial compressive loading is considered as presented in Fig. 2. A polynomial displacement function which was in equation 10 and applied to determine the value of the critical buckling load of the plate.
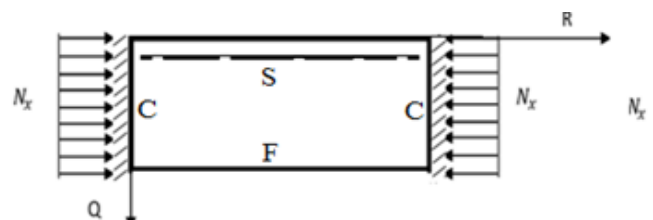

Fig 2. SCFC rectangular plate subjected to uniaxial compressive load.

The boundary conditions of the plate in Fig. 2 are as follows:

R – Direction:

(a) When R = 0, deflection (w) = 0. $\tag{24}$

(b) When R = 0, slope (w') = 0, $\left(ie.\frac{dw}{dR} = 0\right)$ (25)

(c) When R = 1, deflection (w) = 0.   (26)

(d) When R = 1, slope (w') = 0, $\left(ie.\frac{dw}{dR} = 0\right)$ (27)

Q – Direction

(e) When Q = 0, deflection (w) = 0.   (28)

(f) When Q = 0, bending moment = 0, $\left(ie.\frac{d^2w}{dQ^2} = 0\right)$ (29)

(g) When Q = 1, bending moment = 0, $\left(ie.\frac{d^2w}{dQ^2} = 0\right)$ (30)

(h) When Q = 1, shear force = 0, $\left(ie.\frac{d^3w}{dQ^3} = 0\right)$ (31)

(i) When Q = 1, slope (w') = $\frac{2}{3b_5}$, $\left(ie.\frac{dw}{dQ} = \frac{2}{3b_5}\right)$ (32)

Substituting (24) to (32) into the derivatives of w and solving gave the characteristic equations with the following constants:

Substituting equation (24) to (32) into the derivatives of w and solving gave the characteristic equations with the following constants:

$$a_0 = 0;\ a_1 = 0;\ a_2 = 2a_4;\ a_3 = -2a_4 \quad (33)$$

and

$$b_0 = 0;\ b_1 = -\frac{7}{3}b_5;\ b_2 = 0;\ b_3 = -\frac{10b_5}{3};$$
$$b_4 = -\frac{10b_5}{3} \quad (34)$$

Substituting the constants of (33) and (34) into (10) gives:

$$w = (R^2 - 2R^3 + R^4).(7Q - 10Q^3 + 10Q^4 - 3Q^5) \quad (35)$$

Simplifying (35) which satisfying the boundary conditions of (24) to (28) gives:

$$w = a_4(R^2 - 2R^3 + R^4).b_5\left(\frac{7Q}{3} - \frac{10}{3}Q^3 + \frac{10}{3}Q^4 - Q^5\right) \quad (36)$$

That is:

$$w = a_4.b_5(R^2 - 2R^3 + R^4).\left(\frac{7Q}{3} - \frac{10}{3}Q^3 + \frac{10}{3}Q^4 - Q^5\right) \quad (37)$$

where

$$A_1 = a_4 \times b_5 \quad (38)$$
and

$$h = (R^2 - 2R^3 + R^4).\left(\frac{7Q}{3} - \frac{10}{3}Q^3 + \frac{10}{3}Q^4 - Q^5\right) \quad (39)$$

Thus, the polynomial deflection functions after satisfying the boundary conditions is:

$$w = (R^2 - 2R^3 + R^4)\left(\frac{7Q}{3} - \frac{10}{3}Q^3 + \frac{10}{3}Q^4 - Q^5\right).A_1 (40)$$

As such, a numerical values of the stiffness for a SCFS plate were obtained using (24) to (32) by applying the polynomial function as obtained in (39) and their results are presented in Table I.

$$k_{RR} = 0.6709;\ k_{RQ} = 0.0405;\ k_{QQ} = 0.0060;$$
$$k_R = 0.0160;\ k_Q = 0.0034$$

III. RESULTS AND DISCUSSIONS

In this section, a numerical value of the buckling load expression obtained in in the previous section is presented. The non-dimensional value of the critical buckling load for a clamped thick rectangular plate which was simply supported at the first and freely supported at the third edge (SCFC) and subjected to a uniaxial compressive load at varying aspect ratio is presented in Fig. 3 to Fig. 12. This result was obtained by expressing the shape function of the plate in the form of polynomial to obtain the critical buckling load of the plate. A numerical and graphical representation was presented in Fig. 3 to 7 to show the behavior of the element a thick plate's stability at varying thickness and aspect ratio.

The values obtained in Fig. 3 to 7, shows that as the values of critical buckling load increase, the span- thickness ratio increases. This reveals that as the in-plane load on the plate increase and approaches the critical buckling, the failure in a plate structure is a bound to occur; this means that a decrease in the thickness of the plate, increases the chance of failure in a plate structure. Hence, failure tendency in the plate structure can be mitigated by increasing its thickness.

It is also observed in the tables that as the aspect ratio of the plate increases, the value of critical buckling load decreases while as critical buckling load increases as the length to breadth ratio increases. This implies that an increase in plate width increases the chance of failure in a plate structure. It can be deduced that as the in-plane load which will cause the plate to fail by compression increases from zero to critical buckling load, the buckling of the plate exceeds specified elastic limit thereby causing failure in the plate structure. This meant that, the load that causes the plate to deform also causes the plate material to buckle simultaneously.

The result of the present model which analyzed all the stress elements in the plate is considered safer to use to achieve an exact three-dimensional plate analysis using polynomial displacement functions, hence, provides accurate or reliable solution in the analysis of a rectangular plate under SCFS boundary condition. Thus, the present theory obviates the numerical approximations in the thickness direction thereby guaranteed accuracy in the solution of the displacement along the direction of the thickness axis of the plate, hence, a significant lessening of the cost of computation.
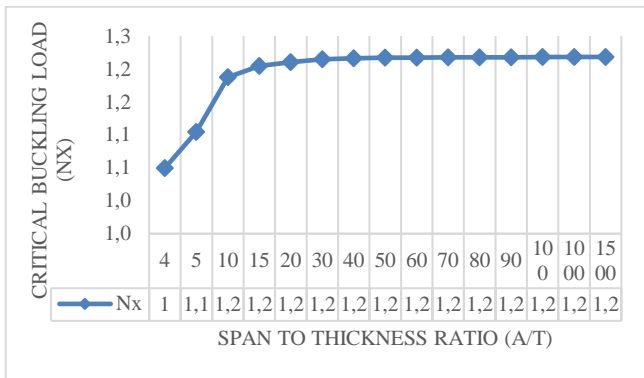
Fig. 3. Critical buckling load (N_x) versus span to thickness ratio (a/t) of a square rectangular plate.

Fig. 4. Critical buckling load (N_x) versus span to thickness ratio (a/t) of a rectangular plate with aspect ratio of 1.5.

Fig. 5. Critical buckling load (N_x) versus span to thickness ratio (a/t) of a rectangular plate with aspect ratio of 2.0.

Fig. 6. Critical buckling load (N_x) versus span to thickness ratio (a/t) of a rectangular plate with aspect ratio of 2.5.

Fig. 7. Critical buckling load (N_x) versus span to thickness ratio (a/t) of a rectangular plate with aspect ratio of 3.0.

Fig. 8. Critical buckling load (N_x) versus span to thickness ratio (a/t) of a rectangular plate with aspect ratio of 3.5.

Fig. 9. Critical buckling load (N_x) versus span to thickness ratio (a/t) of a rectangular plate with aspect ratio of 4.0.

Fig. 10. Critical buckling load (N_x) versus span to thickness ratio (a/t) of a rectangular plate with aspect ratio of 4.5.

Fig. 11. Critical buckling load (N_x) versus span to thickness ratio (a/t) of a rectangular plate with aspect ratio of 5.0.

IV. CONCLUSION

From the result of the analysis of this study, it can be concluded that an increase in plate width increases the chance of failure in a plate structure while the increase in the plate thickness ensures safety in the plate structure subjected to uniaxial compression. Furthermore, it is shown that as the in-plane load which will cause the plate to fail by compression increases from zero to critical buckling load, the buckling of the plate exceeds specified elastic limit thereby causing failure in the plate structure. More, so, the polynomial displacement function developed to give a close form solution, thereby considered more accurate and safer for complete exact three-dimensional thick plate analysis than the polynomial. Its use in the analysis of thick plates will yield almost an exact result. Thus, proof that the 3-D plate theory provides a reliable solution in the stability analysis of plates and can be recommended for analysis of any type of rectangular plate under support condition and load configuration.

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